

A bound to kill the ramification over function fields

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Abstract

Let k be a field of characteristic zero, let X be a geometrically integral k -variety of dimension n and let K be its field of fractions. Under the assumption that K contains all r^{th} roots of unity for an integer r , we prove that, given an element $\alpha \in H^m(K, \mathbb{Z}/r)$, there exist n^2 functions $\{f_i\}_{i=1, \dots, n^2}$ such that α becomes unramified in $L = K(f_1^{1/r}, \dots, f_{n^2}^{1/r})$.

1. Introduction. Let K be a field and let $\alpha \in \text{Br } K$ be an element of order r . In [4], [5], Saltman proved that if K is the function field of a p -adic curve and $(r, p) = 1$, then α becomes trivial over an extension of K of degree r^2 . As a motivation for the question we consider in this paper, let us give a brief sketch of his arguments. Let us assume that r is prime and that K contains all r^{th} roots of unity. In fact, one can see that this case implies the general case. We view K as a function field of a regular, integral two-dimensional scheme X , projective over the spectrum of the ring of integers of a p -adic field. Saltman then proved that one can find two functions $f_1, f_2 \in K$ such that α becomes unramified in $L = K(f_1^{1/r}, f_2^{1/r})$ with respect to any rank one discrete valuation ring centered on X . This is sufficient to conclude, using the classical result that the Brauer group of a regular flat proper (relative) curve over the ring of integers of a p -adic field is trivial (cf. [3], [6]).

Let us consider the case of higher dimensions, that is, assume that K is the field of fractions of an n -dimensional variety X , defined over a field k . Following Saltman's work, given a class $\alpha \in \text{Br } K$, one may wonder if there is a bound N depending only on K , such that we can kill the ramification of α with N functions. Our main result (cf. theorem 1) gives an affirmative answer $N = n^2$ for α of order r under the assumption that K contains all r^{th} roots of unity. Our method also works for elements of $H^m(K, \mathbb{Z}/r)$ and not only for $m = 2$.

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2. Statement of the main result. Let k be a field of characteristic zero. For L a function field over k containing all r^{th} roots of unity we fix an isomorphism $\mu_r \xrightarrow{\sim} \mathbb{Z}/r$ of $\text{Gal}(\bar{K}/K)$ -modules and we write

$$H_{nr}^m(L/k, \mathbb{Z}/r) = \bigcap_A \text{Ker}[H^m(L, \mathbb{Z}/r) \xrightarrow{\partial_A} H^{m-1}(k_A, \mathbb{Z}/r)],$$

where A runs through all discrete valuation rings of rank one with $k \subset A$ and fraction field L . We denote by k_A the residue field of A and by ∂_A the residue map.

Theorem 1. *Let k be a field of characteristic zero. Let X be an integral k -variety of dimension n and let K be its field of fractions. Let r be an integer and assume that K contains all r^{th} roots of unity. Let α be an element of $H^m(K, \mathbb{Z}/r)$. There exist n^2 functions $\{f_i\}_{i=1, \dots, n^2}$ such that α becomes unramified over $L = K(f_1^{1/r}, \dots, f_{n^2}^{1/r})$, that is, we have $\alpha_L \in H_{nr}^m(L/k, \mathbb{Z}/r)$.*

We first prove two lemmas.

3. Local description. In the case of dimension two, the following statement is due to Saltman (cf. [4] 1.2).

Lemma 2. *Let k be an infinite field. Let A be a local ring of a smooth k -variety and let K be its field of fractions. Let r be an integer prime to characteristic of k . Assume that K contains all r^{th} roots of unity and fix an isomorphism $\mu_r \xrightarrow{\sim} \mathbb{Z}/r$ of $\text{Gal}(\bar{K}/K)$ -modules. Let α be an element of $H^m(K, \mathbb{Z}/r)$ ramified only at s_1, \dots, s_h forming a regular subsystem of parameters of the maximal ideal of A . Then*

$$\alpha = \alpha_0 + \sum_{\emptyset \neq I \subset \{1, \dots, h\}} \alpha_I \cup s_I,$$

with $\alpha_0 \in H^m(A, \mathbb{Z}/r)$, $\alpha_I \in H^{m-|I|}(A, \mathbb{Z}/r)$, and $s_I = \cup_{i \in I} (s_i)$, where we denote by (s_i) the class of s_i in $H^1(K, \mathbb{Z}/r) \simeq K^*/K^{*r}$.

Proof. We proceed by induction on h and m . Assume first $h = 1$. For A a local ring of a smooth k -variety, with field of fractions K and for $Y = \text{Spec } A$, we have an exact sequence due to Bloch and Ogus (cf. [1] 2.2.2)

$$0 \rightarrow H^m(A, \mathbb{Z}/r) \rightarrow H^m(K, \mathbb{Z}/r) \rightarrow \coprod_{x \in Y^{(1)}} H^{m-1}(\kappa(x), \mathbb{Z}/r) \rightarrow \coprod_{x \in Y^{(2)}} H^{m-2}(\kappa(x), \mathbb{Z}/r) \rightarrow \dots \quad (1)$$

where the maps are induced by the residues. Denote by $K(A/s_1)$ the field of fractions of A/s_1 . As α is ramified only at s_1 , we see from the sequence (1) that $\partial_{s_1}(\alpha) \in H^{m-1}(K(A/s_1), \mathbb{Z}/r)$ is unramified. Hence, from the sequence (1) for A/s_1 , it comes from an element of $H^{m-1}(A/s_1, \mathbb{Z}/r)$. From Levine's conjecture (generalizing Bloch-Kato's conjecture proved by Rost and Voevodsky), proved by Kerz [2] 1.2, any element of $H^{m-1}(A/s_1, \mathbb{Z}/r)$ is a sum of cup products of units in A/s_1 . In particular, any element of $H^{m-1}(A/s_1, \mathbb{Z}/r)$ lifts to A : there exists an element $\alpha_1 \in H^{m-1}(A, \mathbb{Z}/r)$ such that $\bar{\alpha}_1 = \partial_{s_1}(\alpha)$. Hence $\alpha - \alpha_1 \cup (s_1)$ is unramified, so it comes from $\alpha_0 \in H^m(A, \mathbb{Z}/r)$, by (1) again.

If $m = 1$, we have $\alpha = (s)$ for s a function in K and the result follows from the decomposition $s = u \prod_i s_i^{t_i}$ with $t_i \in \mathbb{Z}$ and $u \in A^*$.

Next, we assume the assertion for $(m-1, h-1)$ and $(m, h-1)$ and we prove it for (m, h) . From the sequence (1), $\partial_{s_1}(\alpha) \in H^{m-1}(K(A/s_1), \mathbb{Z}/r)$ is ramified only at $\bar{s}_2, \dots, \bar{s}_h$ where we denote by \bar{s}_i the image of s_i in A/s_1 . By induction, $\partial_{s_1}(\alpha) = \bar{\alpha}_1 + \sum_{\emptyset \neq I \subset \{2, \dots, h\}} \bar{\alpha}_I \cup \bar{s}_I$, where $\bar{\alpha}_1 \in H^{m-1}(A/s_1, \mathbb{Z}/r)$, $\bar{\alpha}_I \in H^{m-1-|I|}(A/s_1, \mathbb{Z}/r)$, and $\bar{s}_I = \cup_{i \in I} (\bar{s}_i)$. As before, we deduce from [2] 1.2 that all the $\bar{\alpha}_I$ and $\bar{\alpha}_1$ are sums of cup products of units in A/s_1 and so we can lift them to α_I (resp. to α_1) on A . Now the element $\alpha - (\alpha_1 + \sum_{\emptyset \neq I \subset \{2, \dots, h\}} \alpha_I \cup s_I) \cup (s_1)$ is ramified only at s_2, \dots, s_h and the lemma follows by induction. \square

4. Divisor decomposition.

Lemma 3. *Let k be a field of characteristic zero and let X be a smooth projective k -variety of dimension n . Let D be a divisor on X . There exists a sequence of blowing-ups $f : X' \rightarrow X$ such that the support of the total transform f^*D is a simple normal crossing divisor which can be expressed a union of n regular (but not necessarily connected) divisors of X' .*

Proof. By Hironaka, we may assume that $\text{Supp}(D)$ is a simple normal crossing divisor, which means that any irreducible component of $\text{Supp}(D)$ is smooth and that the fiber product over X of any c components of $\text{Supp}(D)$ is smooth and of codimension c . Let $G = (V, E)$ be the dual graph of D :

- the vertices of V correspond to irreducible components D_1, \dots, D_N of D
- the edge (D_i, D_j) is in E if the intersection $D_i \cap D_j$ is nonempty.

We say that we blow-up the edge (D_i, D_j) if we change X by the blow-up of the intersection $D_i \cap D_j$ (with reduced structure) and we change G by the dual graph of the total transform of D , i.e. we add a vertex and corresponding edges. We write again $G = (V, E)$ for the modified graph.

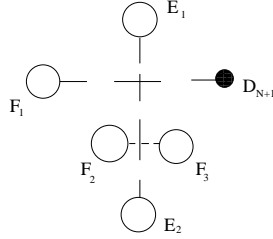
We will show that after a finite sequence of blowing-ups $f : X' \rightarrow X$ of some edges we may color the vertices of G in n colors so that for any edge $AB \in E$ the vertices A and B are of different colors. Then $\text{Supp}(f^*D)$ is a simple normal crossing divisor and we have $\text{Supp}(f^*D) = \bigcup_{i=1}^n F_i$ where F_i is the (disjoint) union of components of f^*D such that the corresponding vertex is of the i^{th} color. Hence F_i are regular and the lemma follows.

If $n = 2$ we may assume, after blowing-ups of some edges, that any cycle in G has even number of edges, which is sufficient to conclude.

Let us now assume that $n \geq 3$. We proceed by induction on the number N of irreducible components of D . If $N \leq n$ the statement is clear. Assume it holds for N . Let D be a divisor with $N + 1$ components. By the induction hypothesis, after

blowing-ups of some edges, we may assume that we may color all but the vertex D_{N+1} of G in n colors as desired. We have $\text{Supp}(D) = \bigcup_{i=1}^n F_i \cup D_{N+1}$ where F_i is the union of components of D of the i^{th} color. If D_{N+1} doesn't intersect F_i for some i we color D_{N+1} in i^{th} color. Hence we may assume that all the intersections $D_{N+1} \cap F_i$ are nonempty. By the same reason, we may assume that the intersection $F_2 \cap F_3$ is nonempty. On the other hand, note that the intersection $\bigcap_i F_i \cap D_{N+1}$ is empty as $\text{Supp}(D)$ is a simple normal crossing divisor. We proceed by the following algorithm:

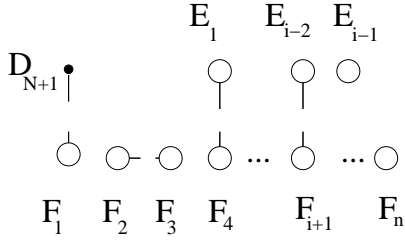
1. We first blow up all the edges $D_j D_{N+1}$ for all the components D_j of F_1 . Let us denote E_1 the union of all the exceptional divisors. This union is disjoint as the components of F_1 do not intersect. Note that $E_1 \cap F_2 \cap \dots \cap F_n = \emptyset$. Otherwise, we get a point in the intersection $\bigcap_i F_i \cap D_{N+1}$ by projection. Moreover, there are no more edges between (the components of) F_1 and D_{N+1} as the strict transforms of the corresponding divisors do not intersect.
2. Next, we blow up all the edges between F_2 and F_3 and we call E_2 the (disjoint) union of all new exceptional divisors. Again, we have no more edges between F_2 and F_3 and also $E_2 \cap E_1 \cap F_4 \cap \dots \cap F_n = \emptyset$ (or $E_2 \cap E_1$ is empty if $n = 3$).
3. If $n = 3$ we have the following picture:



Here and in what follows the punctured line (for example, $F_2 F_3$) means that there are no edges between components of corresponding groups (e.g. no edges between elements of $F_2 \cup F_3$).

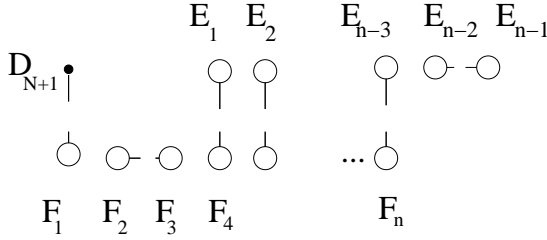
We color (all the vertices from) F_1 and D_{N+1} in red, E_1 and E_2 in green and F_2 and F_3 in blue and this terminates the algorithm.

4. Assume that $n \geq 4$. We proceed until we get the group of exceptional divisors E_{n-1} and then we go to step 6. Suppose $3 \leq i \leq n-1$ and we constructed E_{i-2} and E_{i-1} but no E_i . Suppose there are some edges between E_{i-2} and F_{i+1} , otherwise we go to step 5. We blow up all these edges and we call E_i the (disjoint) union of all new exceptional divisors. We get no more edges between E_{i-2} and F_{i+1} and also $E_i \cap E_{i-1} \cap F_{i+2} \cap \dots \cap F_n = \emptyset$.
5. If there are no edges between E_{i-2} and F_{i+1} , we have the following picture:



We color F_1 and D_{N+1} in the first color, F_2 and F_3 in the second color, E_1 and F_4 in the third, \dots , E_{i-2} and F_{i+1} in the i^{th} -color, E_{i-1} in color $i + 1$, and, finally, F_{i+2}, \dots, F_n in colors $i + 2, \dots, n$ respectively.

6. At this step, we have the following picture:



Moreover, $E_{n-1} \cap E_{n-2} = \emptyset$ by construction. We color F_1 and D_{N+1} in the first color, F_2 and F_3 in the second color, E_1 and F_4 in the third, \dots , E_{n-3} and F_n in color $n - 1$, E_{n-1} and E_{n-2} in color n . This terminates the algorithm. \square

5. Proof of theorem 1. By resolution of singularities, we may assume that X is smooth. By lemma 3, we may assume that the ramification divisor $D = \text{ram}(\alpha)$ is a simple normal crossing divisor whose support is a union of n regular divisors: $\text{Supp } D = \bigcup_{i=1}^n D_i$.

For two divisors G and G' on X , with $G = \sum_{i=1}^q G_i$ where the G_i are irreducible divisors, we say that G' is in *general position* with G if the support of G' contains no generic point of any intersection $\bigcap_{i \in I} G_i$ for $I \subset \{1, \dots, q\}$.

By a semilocal argument, we successively choose functions $f_1^j \in K$, $j = 1, \dots, n$, then $f_2^j \in K$, $j = 1, \dots, n, \dots$, and then $f_n^j \in K$, $j = 1, \dots, n$, such that

$$\text{div}_X(f_i^j) = D_i + E_i^j$$

where E_i^j are in general position with $D \cup \bigcup_{j' < j} \text{Supp}(E_i^{j'})$.

We claim that with this choice of n^2 functions α_L is unramified. Let v be a discrete valuation on L and let $x \in X$ be the point where the discrete valuation ring R of v is centered. We may assume that $x \in \text{Supp } D$, otherwise α is already unramified at v . From the construction, for any i , $D \cap \bigcap_{j=1}^n E_i^j = \emptyset$. Hence for any $1 \leq i \leq n$ we can find j_i such that $x \notin E_i^{j_i}$, which means that the corresponding local parameter s_i of D_i at x is an r^{th} power in $K((f_i^{j_i})^{1/r})$. Now the theorem follows from lemma 2, as any s_I from the lemma is a cup product of r^{th} powers on L . \square

Remark 4. The bound n^2 is not sharp. For example, for $n = 3$ one can kill all the ramification with four functions. Let us write $\text{ram}(\alpha) = D_1 \cup D_2 \cup D_3$ as in lemma 3. As in the proof of the theorem above, we take $f_i \in K$, $i = 1, \dots, 4$, such that

$$\begin{aligned} \text{div}(f_1) &= D_1 + D_2 + D_3 + E_1; \\ \text{div}(f_2) &= D_1 + D_2 + E_2; \\ \text{div}(f_3) &= D_2 + D_3 + E_3; \\ \text{div}(f_4) &= D_1 + 2D_2 + D_3 + E_4. \end{aligned}$$

and each E_i is in general position with $\text{ram}(\alpha) \cup \bigcup_{i' < i} \text{Supp}(E_{i'})$. Let x be a center of a valuation v on $L = K(f_i^{1/r})_{i=1, \dots, 4}$. We may assume that $x \in \text{ram}(\alpha)$. It is sufficient to see that if $x \in D_i$ then a local parameter of D_i at x can be expressed as a product of powers of the functions f_i .

1. If $x \in X^{(1)}$ then x lies on only one component D_i , which is thus defined by f_1 .
2. If $x \in X^{(2)}$ and if x lies on two components D_i and D_j , then $\frac{f_1}{f_3}$ defines D_1 , $\frac{f_2 f_3}{f_1}$ defines D_2 , $\frac{f_1}{f_2}$ defines D_3 . If x lies on only one component D_i , then, by construction, $x \notin E_{i_1} \cup E_{i_2}$ for at least two indexes $1 \leq i_1 < i_2 \leq 3$. By construction, D_i is then defined at x by at least one among the functions f_{i_1} and f_{i_2} .
3. Suppose that x is a closed point of X . If $x \in D_1 \cap D_2 \cap D_3$, we use the same formulas as in the previous case. Next, suppose that x lies on only two components of $\text{ram}(\alpha)$. Consider the case $x \in D_1 \cap D_2$, the other cases are similar. By construction, x lies on at most one component among E_1, E_2, E_3 . Hence we see that if $x \notin E_2 \cup E_3$ (resp. $x \notin E_1 \cup E_3$, resp. $x \notin E_1 \cup E_2$) then D_1 is defined by $\frac{f_2}{f_3}$ and D_2 is defined by f_3 (resp. by $\frac{f_1}{f_3}$ and by f_3 , resp. by $\frac{f_1^2}{f_4}$ and by $\frac{f_4}{f_1}$).

The last case is when x lies on only one component of $\text{ram}(\alpha)$. Consider the case $x \in D_1$, the other cases are similar. Then $x \notin E_1 \cap E_2 \cap E_4$ by construction. Then D_1 is defined by f_1 (resp. by f_2, f_4) if x does not lie on E_1 (resp. on E_2, E_4).

Remark 5. By the same arguments as in the previous remark, if r is prime to 2 and 3 and if $n = 3$, one can kill all the ramification with three functions f_1, f_2, f_3 , such that

$$\begin{aligned} \text{div}(f_1) &= D_1 + 3D_2 + 3D_3 + E_1; \\ \text{div}(f_2) &= D_1 + 2D_2 + D_3 + E_2; \\ \text{div}(f_3) &= D_1 + D_2 + 2D_3 + E_3. \end{aligned}$$

and each E_i is in general position with $\text{ram}(\alpha) \cup \bigcup_{i' < i} \text{Supp}(E_{i'})$.

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